

# Two Similarity Reductions and New Solutions for the Generalized Variable-Coefficient KdV Equation by Using Symmetry Group Method

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## Abstract

In this paper, a generalized variable-coefficient KdV equation (vcKdV) arising in fluid mechanics, plasma physics and ocean dynamics is investigated by using symmetry group analysis. Two basic generators are determined, and for every generator in the optimal system the admissible forms of the coefficients and the corresponding reduced ordinary differential equation are obtained. The search for solutions to those reduced ordinary differential equations yields new exact solutions for the generalized vcKdV equation.

**Key words:** symmetry group method, generalized variable-coefficients KdV equation (vcKdV), Tanh-function method.

## 1 Introduction

In 1895, Korteweg and de Vries developed a nonlinear partial differential equation to model the propagation of shallow water waves [1]. This famous classical equation is known simply as the KdV equation. Recently, the KdV equation have been derived and modified in many different branches of science and engineering including the pulse-width modulation [2], mass transports in a chemical response theory [3], dust acoustic solitary structures in magnetized dusty plasmas [4] and nonlinear long dynamo waves observed in the Sun [5]. However, the high-order nonlinear terms must be taken into account in some complicated situations like at the critical density or in the vicinity of the critical velocity [6–8]. The modified KdV (mKdV)-typed equation, on the other hand, has recently been used, e.g., to model the dust-ion-acoustic waves in such cosmic environments as those in the supernova shells and Saturn’s F-ring [9].

In many real physical backgrounds, the variable-coefficient nonlinear evolution equations often can provide more powerful and realistic models than their constant-coefficient counterparts when the inhomogeneities of media and nonuniformities of boundaries are considered [10-22]. Thereby, some inhomogeneous KdV models with the time-dependent coefficients have been derived to describe a variety of interesting and significant phenomena in fluid mechanics, ocean dynamics and plasma physics, as follows:

In fluid mechanics, by considering the blood as incompressible fluid, the evolution equation is obtained as a variable-coefficient mKdV (vcMKdV) equation [14]

$$U_\tau + \mu_4 U^2 U_\zeta + \mu_2 U_{\zeta\zeta\zeta} + \mu_3 h_2(\tau) U_\zeta = 0, \quad (1)$$

where  $\mu_2, \mu_3$  and  $\mu_4$  are some constants describing the material properties of the tube and the variable  $U$  is the dynamical radial displacement of the elastic tube.

In ocean dynamics, the pycnocline lies midway between the sea bed and surface, the following vcMKdV model is obtained [8]

$$u_t + a(t) u u_x + k u^2 u_x + u_{xxx} = 0, \quad (2)$$

where  $a(t)$  depends on the pycnocline location and  $k$  describes the internal waves in a stratified ocean, as observed on the northwest shelf of Australia [23] and in the Gotland deep of the Baltic Sea [24]. Also, in an inhomogeneous two-layer shallow liquid, the governing equation modelling the long wave propagation is the extended KdV model with variable coefficients [25-26]

$$u_t - 6\alpha(t) u u_x - 6\gamma u^2 u_x + \theta(t) u_{xxx} = 0, \quad (3)$$

where  $u(x, t)$  is proportional to the elevation of the interface between two layers,  $\alpha(t)$  and  $\theta(t)$  imply that the ratio of the depths of two layers depend on the coordinate “ $t$ ” [27].

In plasma physics, the investigation on the dynamics hidden in the plasma sheath transition layer and inner sheath layer gave a perturbed mKdV model [28],

$$\psi_\tau + 6\psi \psi_\eta + \sigma \psi^2 \psi_\eta - \psi_{\eta\eta\eta} + h(\tau) \psi_\eta = 0, \quad (4)$$

where  $\sigma$  is a constant and  $h(\tau)$  is an analytic function.

For the importance of the variable-coefficient KdV-type equations, we will use the symmetry group analysis to study a generalized variable-coefficient KdV equation (vcKdV) [29]

$$u_t + g_1 u_{xxx} + (g_2 u^3 + g_3 u^2 + g_4 u + g_5) u_x + g_6 u + g_7 = 0, \quad (5)$$

where  $g_i = g_i(t)$ ,  $i = 1, \dots, 7$ , are arbitrary smooth functions of  $t$ . Equation (5) includes considerably interesting equations like equations (1-4). When,  $g_2 = 0$ , equation (5) is reduced to that in Ref. [30] derived by considering the time-dependent basic flow and boundary conditions from the well-known Euler equation with an earth rotation term. Also, if  $g_2 = g_7(t) = 0$ , equation (5) becomes the generalized Gardner equation with variable coefficients which arising

in nonlinear lattice, plasma physics and ocean dynamics [31]. Although the KdV equation has been extensively studied for a long time via various methods, yet the vcKdV equation has not been well handled as far as we known. Recently, Qi Wang in [29] used the semi-inverse method to obtain the variational principle for it but no solutions obtained. Therefore, the most important target for this paper is obtaining new exact solutions for equation (5) under some constraints among the variable coefficients by using the symmetry group analysis.

## 2 Symmetry method

We briefly outlined Steinberg's similarity method of finding explicit solutions for both linear and non-linear partial differential equations [32 – 37]. The method based on finding the symmetries of the differential equations is as follows: Suppose that the differential operator  $L$  can be written in the form

$$L(u) = \frac{\partial^p u}{\partial t^p} - H(u), \quad (6)$$

where  $u = u(t, x)$  and  $H$  may depend on  $t, x, u$  and any derivative of  $u$  as long the derivative of  $u$  dose not contain more than  $(p - 1)$ ,  $t$  derivatives. We will consider the symmetry operator (called infinitesimal symmetry), being quasi-linear partial differential operator of first-order, in the form of

$$S(u) = A(t, x, u) \frac{\partial u}{\partial t} + \sum_{i=1}^n B_i(t, x, u) \frac{\partial u}{\partial x_i} + C(t, x, u), \quad (7)$$

and define the Frèchet derivative of  $L(u)$  by

$$F(L, u, v) = \frac{d}{d\varepsilon} L(u + \varepsilon v)|_{\varepsilon=0} \quad (8)$$

With these definitions, we will conduct the following steps:

- (i) Compute  $F(L, u, v)$ ;
- (ii) Compute  $F(L, u, S(u))$ ;
- (iii) Substitute  $H(u)$  for  $(\frac{\partial^p u}{\partial t^p})$  in  $F(L, u, S(u))$ ;
- (iv) Set this expression to zero and perform a polynomial expansion;
- (v) Solve the resulting partial differential equations. Once this system of partial differential equations is solved for the coefficients of  $S(u)$ , Eq. (6) can be used to obtain the functional form of the solutions.

### 3 Determination of the symmetries

The vcKdV equation can be expressed as

$$L(u) = u_t + g_1 u_{xxx} + (g_2 u^3 + g_3 u^2 + g_4 u + g_5) u_x + g_6 u + g_7 = 0. \quad (9)$$

In order to find the symmetries of Eq. (9), we set the following symmetry operator

$$S(u) = A(x, t, u) u_t + B(x, t, u) u_x + C(x, t, u). \quad (10)$$

Calculating the Fréchet derivative  $F(L, u, v)$  of  $L(u)$  in the direction of  $v$ , given by Eq. (8), and replacing  $v$  by  $S(u)$  in  $F$ , we get

$$\begin{aligned} F(L, u, S(u)) &= S_t + g_1 S_{xxx} + 3g_2 u^2 u_x S + g_2 u^3 S_x + g_3 u^2 S_x \\ &\quad + 2g_3 u u_x S + g_4 u_x S + g_4 u S_x + g_5 S_x + g_6 S + g_7. \end{aligned} \quad (11)$$

Substituting the values of different derivatives of  $S(u)$  in  $F$  with the aid of Maple program, we get a polynomial expansion in  $u_x, u_t, u_y, u_x u_t, \dots$ , etc. On making use of Eq. (9) in the polynomial expression for  $F$ , rearranging terms of various powers of derivatives of  $u$  and equating them to zero, we obtain

$$\begin{aligned} A_x &= A_u = B_u = C_u u = C_{xu} = 0, \\ 3g_1 B_x - (Ag_1)_t &= 0, \\ C_t - (Ag_7)_t - (Ag_6)_t u + g_1 C_{xxx} + g_2 C_x u^3 + g_3 C_x u^2 + g_4 C_x u \\ &\quad + g_5 C_x - g_6 C_u u - g_7 C_u + g_6 C = 0, \\ g_2 B_x u^3 + g_5 B_x - (Ag_5)_t - (Ag_2)_t u^3 - (Ag_3)_t u^2 + 2g_3 u C + B_t + g_4 C \\ &\quad + 3g_2 u^2 C - (Ag_4)_t u + g_4 B_x u + g_3 B_x u^2 = 0. \end{aligned} \quad (12)$$

On solving system (12), the infinitesimal  $A, B$  and  $C$  in the above equations are:

$$\begin{aligned} A &= \frac{1}{\Gamma'(t)} [3c_1 \Gamma(t) + c_2], \quad \frac{d\Gamma(t)}{dt} = g_1(t), \\ B &= c_1 x + c_3, \\ C &= c_4 u + c_5, \end{aligned} \quad (13)$$

where  $c_i, i = 1, 2, \dots, 5$ , are arbitrary constants. The functions  $g_i = g_i(t), i = 1, 2, \dots, 7$ , are governed by the following equations:

$$\begin{aligned} (Ag_2)_t - (c_1 + 3c_4) g_2 &= 0, \\ (Ag_3)_t - (c_1 + 2c_4) g_3 - 3c_5 g_2 &= 0, \\ (Ag_4)_t - (c_1 + c_4) g_4 - 2c_5 g_3 &= 0, \\ (Ag_5)_t - c_1 g_5 + c_5 g_4 &= 0, \\ (Ag_6)_t &= 0, \\ (Ag_7)_t + c_4 g_7 + c_5 g_6 &= 0. \end{aligned} \quad (14)$$

The symmetry Lie algebra  $L^5$  of Eq. (5) is generated by the operators

$$\begin{aligned}
V_1 &= \frac{3\Gamma(t)}{\Gamma'(t)} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\
V_2 &= \frac{1}{\Gamma'(t)} \frac{\partial}{\partial t}, \\
V_3 &= \frac{\partial}{\partial x}, \\
V_4 &= u \frac{\partial}{\partial u}, \\
V_5 &= \frac{\partial}{\partial u}.
\end{aligned} \tag{15}$$

where  $L^5$  is the direct sum of  $V_1, \dots, V_5$  and the commutator table of it is given by

	$V_1$	$V_2$	$V_3$	$V_4$	$V_5$
$V_1$	0	$-3V_2$	$-V_3$	0	0
$V_2$	$3V_2$	0	0	0	0
$V_3$	$V_3$	0	0	0	0
$V_4$	0	0	0	0	$-V_5$
$V_5$	0	0	0	$V_5$	0

## 4 One-dimensional optimal system of $L^5$

We need to search a one-dimensional optimal system of  $L^5$  by considering a general element of  $L^5$ ,  $V = \sum_{i=1}^5 a_i V_i$ , and checking whether  $V$  can be mapped to a new element  $V^*$  under the general adjoint transformation  $Ad(\exp(\varepsilon V_i)) V_j = V_j - \varepsilon [V_i, V_j] + \frac{\varepsilon^2}{2} [V_i, [V_i, V_j]]$ , so as to simplify it as much as possible.

The adjoint table of $L^5$					
	$V_1$	$V_2$	$V_3$	$V_4$	$V_5$
$V_1$	$V_1$	$\exp(3\varepsilon) V_2$	$\exp(\varepsilon) V_3$	$V_4$	$V_5$
$V_2$	$V_1 - 3\varepsilon V_2$	$V_2$	$V_3$	$V_4$	$V_5$
$V_3$	$V_1 - \varepsilon V_3$	$V_2$	$V_3$	$V_4$	$V_5$
$V_4$	$V_1$	$V_2$	$V_3$	$V_4$	$\exp(\varepsilon) V_5$
$V_5$	$V_1$	$V_2$	$V_3$	$V_4 - \varepsilon V_5$	$V_5$

Following [38], we can deduce the following basic fields which form an optimal system for the coupled KdV system,

- (i)  $V_1 + k_1 V_4$ ,
- (ii)  $V_2 + k_2 V_3 + k_3 V_4$ ,
- (iii)  $V_3 + k_4 V_5$ ,

(iv)  $V_4$ ,  
(v)  $V_5$ ,

where  $k_i, i = 1, \dots, 4$  are arbitrary constants. Because cases (iii), (iv) and (v) give trivial reductions therefore, we will discuss the first and the second cases only.

In order to obtain the invariant transformation in each of the above cases, we write the characteristic equation in the form

$$\frac{dt}{A(x, t, u)} = \frac{dx}{B(x, t, u)} = \frac{-du}{C(x, t, u)}. \quad (16)$$

## 5 Reductions and exact solutions

In this section, the primary focus is on the reductions associated with the two vector fields (i) and (ii), and the admissible forms of the coefficients by solving Eq. (16) and the system of integrability conditions (14)

### **Generator (I)**

The generator (i) in the optimal system defines the similarity variable and the similarity solution as follows:

$$\begin{aligned} \zeta &= (x + k_1) \Gamma(t)^{-\frac{1}{3}}, \\ u(x, t) &= F(\zeta) \Gamma(t)^{-\frac{k_1}{3}}, \end{aligned} \quad (17)$$

and the coefficients are given as:

$$\begin{aligned} g_2(t) &= \frac{n_1}{3} \Gamma'(t) \Gamma(t)^{-\frac{2}{3} + k_1}, \\ g_3(t) &= \frac{n_2}{3} \Gamma'(t) \Gamma(t)^{\frac{2}{3}(k_1 - 1)}, \\ g_4(t) &= \frac{n_3}{3} \Gamma'(t) \Gamma(t)^{\frac{1}{3}(k_1 - 2)}, \\ g_5(t) &= \frac{n_4}{3} \Gamma'(t) \Gamma(t)^{-\frac{2}{3}}, \\ g_6(t) &= \frac{n_5}{3} \Gamma'(t) \Gamma(t)^{-1}, \\ g_7(t) &= \frac{n_6}{3} \Gamma'(t) \Gamma(t)^{-(1 + \frac{k_1}{3})}, \end{aligned} \quad (18)$$

where  $n_1, \dots, n_6$  are arbitrary constants. Using the similarity variable, the forms of the similarity solution and the coefficient functions, the vcKdV is reduced to the following ordinary differential equation.

$$3F''' + n_1 F^3 F' + n_2 F^2 F' + n_3 F F' + (n_4 - \zeta) F' + (n_5 - k_1) F + n_6 = 0. \quad (19)$$

To solve Eq. (19), we seek a special solution in the form of

$$F = A_0 + A_1 \zeta^{-\frac{2}{3}}, \quad (20)$$

where  $A_0$  and  $A_1$  are arbitrary constants to be determined. Substituting Eq. (20) into Eq. (19) and equating the coefficients of different powers of  $\zeta$  to zero, we get a system of algebraic equations, solutions of which give rise to the relations on the constants as

$$A_0 = \frac{3n_6}{2}, \quad A_1 = \left( \frac{60n_6}{n_2} \right)^{\frac{1}{3}}, \quad n_1 = -\frac{2n_2}{9n_6}, \quad n_3 = -\frac{3}{2}n_2n_6, \quad n_4 = \frac{3}{4}n_2n_6^2, \quad k_1 = \frac{2}{3} + n_5. \quad (21)$$

Finally, we get the following exact solution for the vcKdV

$$u_1(x, t) = \Gamma(t)^{-\frac{(\frac{2}{3} + n_5)}{3}} \left( \frac{3n_6}{2} + \left( \frac{60n_6}{n_2} \right)^{\frac{1}{3}} \left( \left[ x + \frac{2}{3} + n_5 \right] \Gamma(t)^{-\frac{1}{3}} \right)^{-\frac{2}{3}} \right) \quad (22)$$

### **Generator (II)**

Corresponding to this generator, the associated similarity variable and similarity solution are given by

$$\begin{aligned} \zeta &= k_2 \Gamma(t) - x, \\ u(x, t) &= F(\zeta) e^{-k_3 \Gamma(t)}, \end{aligned} \quad (23)$$

and the coefficient functions are given by

$$\begin{aligned} g_2(t) &= m_1 \Gamma'(t) e^{3k_3 \Gamma(t)}, \\ g_3(t) &= m_2 \Gamma'(t) e^{2k_3 \Gamma(t)}, \\ g_4(t) &= m_3 \Gamma'(t) e^{k_3 \Gamma(t)}, \\ g_5(t) &= m_4 \Gamma'(t), \\ g_6(t) &= m_5 \Gamma'(t), \\ g_7(t) &= m_6 \Gamma'(t) e^{-k_3 \Gamma(t)}, \end{aligned} \quad (24)$$

where  $m_1, \dots, m_6$  are arbitrary constants.

The reduced ordinary differential equation is

$$F''' + m_1 F^3 F' + m_2 F^2 F' + m_3 F F' + (m_4 - k_3) F' + (k_4 - m_5) F + m_6 = 0. \quad (25)$$

Herein, we apply the modified extended tanh function method [39–40] to obtain exact solitary wave solutions to Eq. (25). Let us assume that Eq. (25) has a solution in the form

$$F(\zeta) = A_0 + \sum_{i=1}^N A_i \phi^i + B_i \phi^{-i}, \quad (26)$$

where  $\phi(\zeta)$  is a solution of the following Riccati equation

$$\phi' = r + \phi^2, \quad (27)$$

which has the following solutions

$$\begin{aligned}
\phi(\zeta) &= -\sqrt{-r} \tanh(\sqrt{-r}\zeta), & r < 0, \\
\phi(\zeta) &= -\sqrt{-r} \coth(\sqrt{-r}\zeta), & r < 0, \\
\phi(\zeta) &= \sqrt{r} \tanh(\sqrt{r}\zeta), & r > 0, \\
\phi(\zeta) &= -\sqrt{r} \coth(\sqrt{r}\zeta), & r > 0, \\
\phi(\zeta) &= -\frac{1}{\zeta}, & r = 0.
\end{aligned} \tag{28}$$

Substitute (26) into Eq. (25) and by balancing the linear term with the greatest nonlinear term we get

$$N = \frac{2}{3} \tag{29}$$

Therefore

$$F(\zeta) = A_0 + A_1 \phi^{\frac{2}{3}} + B_1 \phi^{\frac{-2}{3}}. \tag{30}$$

Substituting (30) into Eq. (25) and equating the powers of  $\phi^j, j = 0, -\frac{11}{3}, -3, -\frac{7}{3}, \dots$  to zero, we obtain a system of algebraic equations. By solving that system with Maple program yields the following solution

$$\begin{aligned}
A_1 &= -2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}}, B_1 = \left( \frac{15}{m_1} \right)^{\frac{2}{3}} \left( \frac{3m_3m_1 - m_2^2}{63m_1} \right), r = \pm \frac{1}{42} \sqrt{\frac{5}{14m_1}} \left( \frac{m_2^2 - 3m_1m_3}{m_1} \right)^{\frac{3}{2}}, \\
A_0 &= -\frac{m_2}{3m_1}, m_6 = 0, k_3 = m_5, k_2 = m_4 + \frac{1}{1323m_1^2} \left( 98m_2^3 - 441m_1m_2m_3 \pm 2\sqrt{70} (m_2^2 - 3m_1m_3)^{\frac{3}{2}} \right).
\end{aligned} \tag{31}$$

Substituting Eq. (31) into Eq. (30), we get

$$F(\zeta) = -\frac{m_2}{3m_1} - 2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}} \phi^{\frac{2}{3}} + \left( \frac{15}{m_1} \right)^{\frac{2}{3}} \left( \frac{3m_3m_1 - m_2^2}{63m_1} \right) \phi^{\frac{-2}{3}}, \tag{32}$$

where  $\phi$  is given by Eq. (28). Substitution of Eq. (32) into Eq. (23) results in

$$u_2(x, t) = e^{-k_3\Gamma(t)} \left[ \frac{-m_2}{3m_1} + 2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}} r^{\frac{1}{3}} \tanh^{\frac{2}{3}}(\sqrt{-r}(k_2\Gamma(t) - x)) - \frac{\left( \frac{15}{m_1} \right)^{\frac{2}{3}} \left( \frac{3m_3m_1 - m_2^2}{63m_1} \right)}{r^{\frac{1}{3}} \tanh^{\frac{2}{3}}(\sqrt{-r}(k_2\Gamma(t) - x))} \right], \tag{33}$$

$$u_3(x, t) = e^{-k_3\Gamma(t)} \left[ \frac{-m_2}{3m_1} + 2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}} r^{\frac{1}{3}} \coth^{\frac{2}{3}}(\sqrt{-r}(k_2\Gamma(t) - x)) + \frac{\left( \frac{15}{m_1} \right)^{\frac{2}{3}} \left( \frac{3m_3m_1 - m_2^2}{63m_1} \right)}{r^{\frac{1}{3}} \coth^{\frac{2}{3}}(\sqrt{-r}(k_2\Gamma(t) - x))} \right], \tag{34}$$

where

$$r = -\frac{1}{42} \sqrt{\frac{5}{14m_1}} \left( \frac{m_2^2 - 3m_1m_3}{m_1} \right)^{\frac{3}{2}} \text{ and } k_2 = m_4 + \frac{1}{1323m_1^2} \left( 98m_2^3 - 441m_1m_2m_3 - 2\sqrt{70} (m_2^2 - 3m_1m_3)^{\frac{3}{2}} \right),$$



$$u_4(x, t) = -e^{-k_3 \Gamma(t)} \left[ \frac{m_2}{3m_1} + 2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}} r^{\frac{1}{3}} \tan^{\frac{2}{3}}(\sqrt{r}(k_2 \Gamma(t) - x)) - \frac{\left( \frac{15}{m_1} \right)^{\frac{2}{3}} \left( \frac{3m_3 m_1 - m_2^2}{63m_1} \right)}{r^{\frac{1}{3}} \tan^{\frac{2}{3}}(\sqrt{r}(k_2 \Gamma(t) - x))} \right], \quad (35)$$

$$u_5(x, t) = -e^{-k_3 \Gamma(t)} \left[ \frac{m_2}{3m_1} + 2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}} r^{\frac{1}{3}} \cot^{\frac{2}{3}}(\sqrt{r}(k_2 \Gamma(t) - x)) - \frac{\left( \frac{15}{m_1} \right)^{\frac{2}{3}} \left( \frac{3m_3 m_1 - m_2^2}{63m_1} \right)}{r^{\frac{1}{3}} \cot^{\frac{2}{3}}(\sqrt{r}(k_2 \Gamma(t) - x))} \right], \quad (36)$$

where

$$r = \frac{1}{42} \sqrt{\frac{5}{14m_1}} \left( \frac{m_2^2 - 3m_1 m_3}{m_1} \right)^{\frac{3}{2}} \text{ and } k_2 = m_4 + \frac{1}{1323m_1^2} (98m_2^3 - 441m_1 m_2 m_3 + 2\sqrt{70} (m_2^2 - 3m_1 m_3)^{\frac{3}{2}}),$$

$$u_6(x, t) = -e^{-k_3 \Gamma(t)} \left( \frac{m_2}{3m_1} + 2 \left( \frac{5}{9m_1} \right)^{\frac{1}{3}} (k_2 \Gamma(t) - x)^{-\frac{2}{3}} \right), \quad (37)$$

where

$$r = 0, k_2 = \frac{-m_2^3}{27m_1^2} + m_4 \text{ and } m_3 = \frac{m_2^2}{3m_1}.$$

## 6 Conclusion

In this paper, we have applied the symmetry group analysis to the vcKdV. This application leads to two nonequivalent generators, for every generator in the optimal system the admissible forms of the coefficients and the corresponding reduced ordinary differential equation are obtained. The search for solutions to those reduced ordinary differential equations using tanh function method has yielded many exact new solutions.

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